Journal of Sound and Vibration (2001) **245**(2), 285–301
doi:10.1006/jsvi.2001.3568, available online at http://www.idealibrary.com on IDEAL®

EXACT, SINGLE EQUATION, CLOSED-FORM SOLUTION OF VIBRATING SYSTEMS WITH PIECEWISE LINEAR SPRINGS

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(*Received* 15 *October* 1999, *and in final form* 17 *January* 2001)

A method is proposed to obtain in a single equation the exact, closed-form displacement response of both free undamped vibrations and the steady state of forced damped vibrations of systems with piecewise linear springs, with continuous or discontinuous force–deflection relations. Since the displacement equations are exact the velocity and acceleration responses are obtained by differentiation. The solutions are complete, they apply for any value of time from zero to infinity. The method is designed to be handled in a computer either in conjunction with a Symbolic Mathematics package, like Mathematica or Maple, or without it. However, for certain relatively simple problems the phase-plane solution may be obtained even in a hand-held calculator. Two examples are presented.

1. INTRODUCTION

In reference [1] a scheme was proposed to represent a "broken line" or discontinuous function in a non-piecewise or unified equation which can be easily integrated. In that article, the scheme was specifically applied to the Clebsch or "pointed brackets" method for beam deflections which thus becomes an automatic process, obviating the use of the pointed brackets altogether. Consequently, it makes the process amenable to use in the computer with, or without, the use of symbolic mathematics software. As a matter of fact, it remains an automatic process even when handled in a hand-held non-programmable calculator.

Furthermore, a "periodizer function" was introduced in reference [2], which together with the unification scheme of reference [1] was used in the linear vibration problem with non-harmonic or discontinuous periodic excitation. This allows the steady state displacement response to be obtained directly, in closed form, without resorting to either Fourier series or the Laplace, or any other, transform. Since the displacement response obtained in this manner is exact, differentiating it yields the velocity, and differentiating the velocity yields the acceleration. This contrasts with the drastic deterioration of convergence, and subsequent required adjustment that results when the Fourier series representation of the displacement is differentiated.

Pursuing further the ideas of references [1, 2], in this paper a single equation is used to represent the force–displacement relation of a piecewise linear spring. This leads to the analytical, single equation, phase-plane solution of the free vibrations of a system made up of such a spring and a mass. Additionally, for both the previously mentioned system and a forced-damped system complete displacement–time solutions, i.e., from time $= 0$ to ∞, are obtained in a single equation each by applying to the piecewise solution the unifying technique of reference [1] and two operators introduced in this paper. The single equations for the complete velocity–time and acceleration–time relations are readily obtained by differentiation.

The basic building block for this methodology is the Heaviside unit step function which can be handled by Symbolic Mathematics packages like Mathematica and Maple, if so desired [3–6]. However, it is not necessary to use any such program since a simple, closed algebraic expression for the Heaviside function has been proposed in reference [1] which permits its inclusion in the equation of motion.

2. MATHEMATICAL DEVICES REQUIRED IN THE SOLUTION

2.1. FUNDAMENTAL LINKING ELEMENT

The methodology proposed here relies on a compact representation of functions which are either discontinuous themselves or have discontinuous derivatives.
Consider the following conventional representation of three well-known functions.

Absolute value or vee:
$$\begin{aligned} V(x, a) &= -x + a, \quad x \leqslant a, \\ V(x, a) &= x - a, \quad x \geqslant a. \end{aligned} \tag{1}$$

Relay or jump:
$$\begin{aligned} J(x, a) &= -1, \quad x \leqslant a, \\ J(x, a) &= 1, \quad x \geqslant a. \end{aligned} \tag{2}$$

Heaviside unit step:
$$\begin{aligned} H(x, a) &= 0, \quad x \leqslant a, \\ H(x, a) &= 1, \quad x \geqslant a. \end{aligned} \tag{3}$$

It is clear that, in this conventional representation, each of these functions is described by four relations, i.e., two equations plus one inequality associated with each equation.
A more compact representation, proposed in reference [1] follows.

Absolute value or vee: $V(x, a) = |x - a| = +\sqrt{(x-a)^2}$. (4)

Relay or jump: $J(x, a) = \dfrac{x - a}{V(x, a)} = \dfrac{V(x, a)}{x - a}$. (5)

Heaviside unit step: $H(x, a) = \frac{1}{2}(1 + J(x, a))$. (6)

It is worth noting that this representation (4)–(6) requires, in each case, only one equality valid from $-\infty$ to $+\infty$ and, therefore, it requires no inequalities at all. Thus, this representation may be incorporated, as is, directly into the equations of motion, if so desired. The fundamental linking element is the Heaviside unit step.

2.2. UNIFIED REPRESENTATION OF BROKEN LINE OR DISCONTINUOUS FUNCTIONS

A unified representation of broken line or discontinuous functions, i.e., piecewise continuous functions, may be built up using the Heaviside unit step as a "switch-on" function and its negative as a "switch off" function [1, 7]. Consequently, a composite function, $g(x)$, may be formed by concatenating the individual functions, $f_1(x)$, $f_2(x)$, $f_3(x), \ldots$ that constitute it. Thus, if function $f_1(x)$ is valid in the interval $x \leqslant x_1$, function $f_2(x)$ is valid in the interval $x_1 \leqslant x \leqslant x_2$, function $f_3(x)$ is valid in the interval $x_2 \leqslant x \leqslant x_3$, and so on, the concatenation is carried out according to the following scheme:

$$g(x) = f_1(x) + H(x, x_1)[-f_1(x) + f_2(x)] + H(x, x_2)[-f_2(x) + f_3(x)] + \cdots. \tag{7}$$

At point $x = x_1$, $H(x, x_1)$ switches off $f_1(x)$ and simultaneously switches on $f_2(x)$. At point $x = x_2$, $H(x, x_2)$ switches off $f_2(x)$ and simultaneously switches on $f_3(x)$ and so on.

2.3. INTEGRATION OF POLYNOMIAL FUNCTIONS

The following formula [1], is used to integrate the composite functions resulting from the unifying procedure previously mentioned:

$$\int H(x, a)(x - a)^N \,\mathrm{d}(x - a) = H(x, a)\frac{(x - a)^{N+1}}{N + 1}, \quad N \neq -1. \tag{8}$$

In Appendix A, equation (8) is established with the aid of an ordinary table of integrals. It is interesting to verify that symbolic integration in the *Mathematica* program, for a specified value of N, yields results in accordance with relation (8). Use of relation (7) and integration with respect to the variable shown in equation (8) results in automatic compliance of analytical continuation.

2.4. PERIODIZER FUNCTION

The periodizer function, proposed in reference [2], is a piecewise linear, periodic (sawtooth) function of the independent variable:

$$p(t, T) = \frac{T}{2} - \frac{T}{\pi}\arctan\left(\cot\frac{\pi}{T}t\right). \tag{9}$$

Figure 1(a) is a direct plot of equation (9).

An interval of a non-periodic function $f(t)$ may be converted into a periodic function by simply substituting the independent variable by the periodizer function, $p(t, T)$, where T is the length of the interval. Thus, $f(p(t, T))$ is the resulting function with the chosen period, T. Furthermore, an interval of a periodic function of period T_f may be converted into a periodic function with the different period, T, the length of the interval.

2.5. REFLECTING AND REPEATING FUNCTION

The reflecting and repeating function proposed here is

$$R(t, T) = \frac{T}{4\pi}\arccos\left(\cos\frac{4\pi}{T}t\right). \tag{10}$$

It is akin to the periodizer function. Replacing the time by R in an expression representing a quarter of a cycle of a periodic alternating function (PAF) has the effect of reflecting the

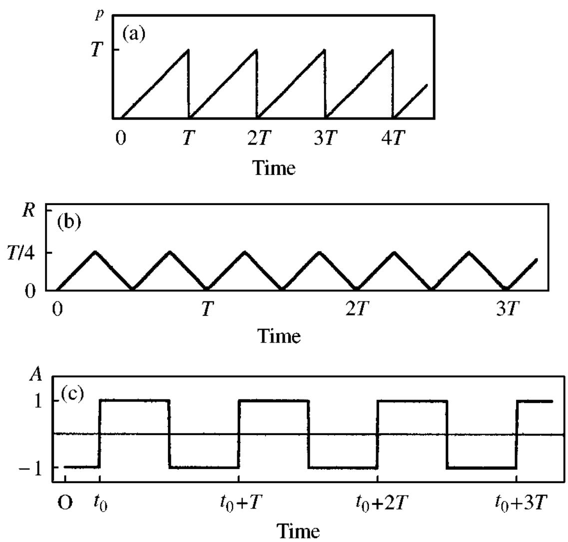


Figure 1. Mathematical devices required in the proposed solutions: (a) periodizer function, direct plot of equation (9); (b) reflecting and repeating function, direct plot of equation (10); (c) alternator function, equation (13).

quarter cycle, thus converting it into a half-cycle which it then repeats indefinitely. Figure 1(b) was plotted directly from equation (10) (after assigning a numerical value to T, of course). The R function should be used only when all four quarters of the cycle have the same shape.

2.6. ALTERNATOR FUNCTION

Let

$$S = \sin(2\pi(t - t_0)/T), \qquad S_a = +\sqrt{S^2} = |S|, \tag{11,12}$$

$t_0 = (T/2\pi)\,\phi$. The alternator function proposed here is

$$A(t, T, t_0) = S/S_a = S_a/S. \tag{13}$$

Figure 1(c) was plotted directly from equation (13). It is essentially a square wave. When a periodic non-alternating function, with period $T/2$, is multiplied by A it is converted into a PAF with period T.

2.7. CONVERSION OF A QUARTER CYCLE INTO A FULL PERIODIC ALTERNATING FUNCTION

If $f_Q(t)$ represents the quarter cycle of a PAF, the full PAF is obtained as

$$f(t) = A(t, T, t_0)f_Q(R(t, T)), \tag{14}$$

where T and t_0 are the period and the phase time, respectively, of $f(t)$. Since A is a square wave the effect of the operation indicated in equation (14) is to multiply by -1 every other cycle of the periodic non-alternating function $f_Q(R(t, T, t_0))$, thus converting into the PAF, $f(t)$.

3. FREE, UNDAMPED VIBRATIONS

3.1. PROCEDURE, PHASE-PLANE SOLUTION

The phase-plane, or velocity versus displacement, solution is obtained in accordance with the following steps: (1) the force–displacement relation of the spring is expressed in a single equation applying equation (7); (2) the force equation of Step 1 is substituted into the differential equation of motion of the system, expressed in terms of the velocity and the displacement; (3) the equation of motion is integrated in accordance with equation (8).

3.1.1. *Example* 1*A*

The broken line, but continuous, force–displacement relation of the spring of the vibrating system of Figure 2 is characterized by a dead zone; Figure 3(a). The phase-plane solution will be obtained in a single equation.

The equation of motion in terms of velocity and displacement is

$$m\dot{x}\,\mathrm{d}\dot{x} + F(x)\,\mathrm{d}x = 0. \tag{15}$$

The force-displacement equation, to represent the relation of Figure 3(a), is established by applying equation (7):

$$F(x) = k(x+e) - H(x,-e)k(x+e) + H(x,e)k(x-e). \tag{16}$$

Substituting equation (16) into equation (15) and integrating in accordance with equation (8) yields

$$m\dot{x}^2 + k[x^2 + 2ex - H(x,-e)(x+e)^2 + H(x,e)(x-e)^2] = C. \tag{17}$$

The constant of integration is evaluated using the initial conditions

$$C = m\dot{x}(0)^2 + k[x(0)^2 + 2ex(0) - H(x,-e)(x(0)+e)^2 + H(x,e)(x(0)-e)^2]. \tag{18}$$

Solving of $\dot{x}$ in equation (17) yields

$$\dot{x} = \pm\sqrt{\frac{k}{m}[-x^2 - 2ex + H(x,-e)(x+e)^2 - H(x,e)(x-e)^2] + C}. \tag{19}$$

Equation (19) is the general phase-plane solution.

For the sake of concreteness the following values will be considered:

$$m = 1\ \mathrm{N\,s^2/cm}, \qquad k = 2\ \mathrm{N/cm}, \qquad e = 3\ \mathrm{cm}, \qquad x(0) = 0\ \mathrm{cm}, \qquad \dot{x}(0) = 2\ \mathrm{cm/s}. \tag{20}$$

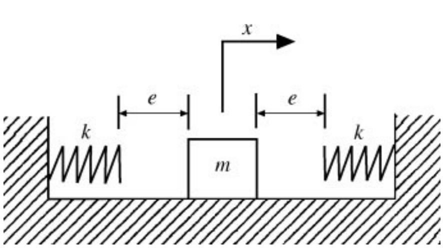
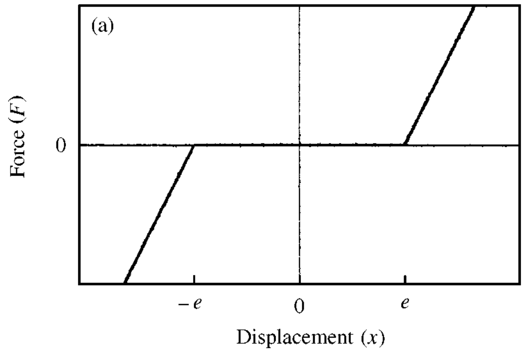


Figure 2. Example 1: mass-piecewise linear spring system. In this particular example the springs exhibit a dead zone condition.



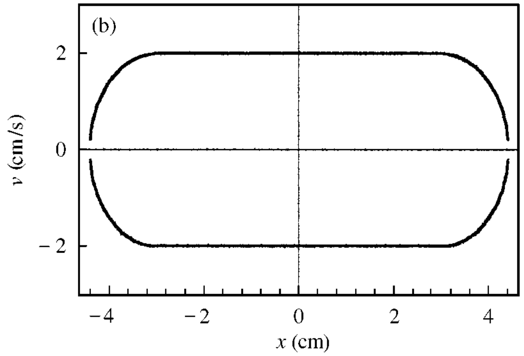


Figure 3. Example 1: (a) force–displacement diagram of the spring, plotted directly from equation (16) after substituting values (20); (b) example 1A; direct plot of the phase-plane solution, equation (21).

Substituting values (20) into equations (18) and (19) yields

$$\dot{x} = \pm\sqrt{-2x^2 - 12x + 2H(x, -3)(x+3)^2 - 2H(x, 3)(x-3)^2 - 14}$$

or in more familiar terms,

$$\dot{x} = \pm\sqrt{-2x^2 - 12x + (x+3)^2 + |x+3|(x+3) - (x-3)^2 - |x-3|(x-3) - 14}. \tag{21}$$

Figure 3(b) is a direct plot of equation (21). Note that, because of the automatic compliance with analytical continuation previously mentioned, only one constant of integration was required, whereas in the customary, non-unified, piecewise solution three constants of integration would have been required. This example may be handled even in a non-programmable calculator and equations (16) and (21) may be plotted in a graphics calculator.

3.2. DISPLACEMENT–TIME SOLUTION, PROCEDURE

It is pertinent to point out that no new approach is proposed to obtain the displacement–time solution. What is done here is simply to fuse into a single equation the various pieces of the conventional piecewise solution, using the proposed mathematical devices and in accordance with the following steps: (1) the piecewise solution of a quarter cycle is unified by the function concatenation procedure, equation (7); (2) the quarter cycle solution is converted into the full periodic alternating solution by the conversion scheme, equation (14), valid from $t = 0$ to ∞.

3.2.1. *Example* 1*B*

The displacement–time solution for example 1A will be obtained. For the sake of expediency, it will be considered from the outset that

$$x(0) = 0, \quad \dot{x}(0) = v(0). \tag{22}$$

The piecewise solution for the first quarter cycle is

$$x_1(t) = v(0)\,t, \quad 0 \leqslant x_1 \leqslant e, \tag{23}$$

$$x_2(t) = e + v(0)\sqrt{\frac{m}{k}}\sin\left[\sqrt{\frac{k}{m}}(t - t_e)\right], \quad x_2 \geqslant e. \tag{24}$$

The time required for the mass to travel from $x = 0$ to $x = e = 3$ cm, the point at which it makes contact with the spring, is

$$t_e = e/v(0). \tag{25}$$

The procedure to construct the single equation closed solution is easily understood by reference to Figure 4. The functions x_1 and x_2 are shown graphically in Figure 4(a), and they are concatenated according to equation (7):

$$x_C(t) = x_1(t) + H(t, t_e)(-x_1(t) + x_2(t)). \tag{26}$$

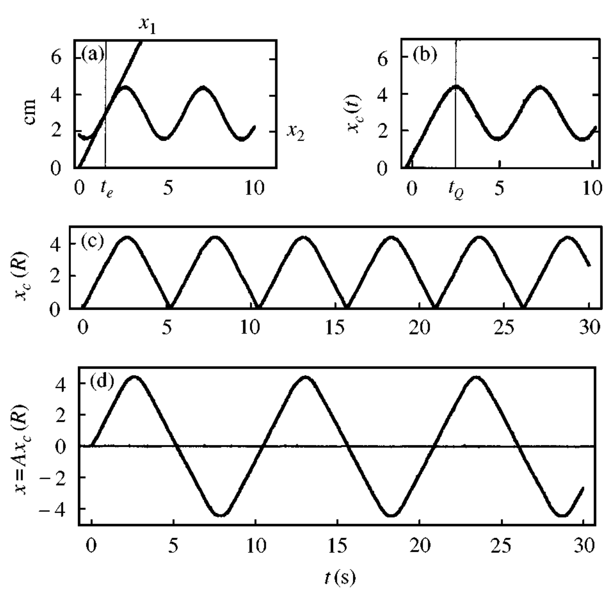


Figure 4. Example 1B. Graphs to illustrate the procedure used to establish the displacement–time solution. (a) Superimposed plots of the functions which make up the solution, x_1 is represented by equation (23) and x_2 by equation (24). (b) The concatenated displacement, equation (26) or (27). (c) Substitution of R instead of t has the effect of first reflecting the concatenated curve from 0 to the end of the quarter cycle about the vertical line $t = t_Q$ to complete one-half of a cycle which it then repeats indefinitely; equation (32). (d) The displacement–time response $x(t)$, equation (33) obtained by multiplying $x_C(R)$ by the alternator function.

The effect of this procedure is easily understood by comparing Figure 4(a) and 4(b). Substituting equations (23) and (24) into equation (26) yields

$$x_C(t) = v(0)t + H(t, t_e)\left[-v(0)t + e + v(0)\sqrt{\frac{m}{k}}\sin\left\{\sqrt{\frac{k}{m}}(t - t_e)\right\}\right]. \tag{27}$$

The concatenated displacement x_C, Figure 4(b), was plotted directly from equation (27).

Using equations (27) and (25), the quarter cycle time, t_Q, may be obtained from the relation

$$\sqrt{\frac{k}{m}}\left(t_Q - \frac{e}{v(0)}\right) = \frac{\pi}{2}. \tag{28}$$

Solving for t_Q and substituting values (20) yields

$$t_Q = 2{\cdot}610720735\ \text{s}, \tag{29}$$

thus the period is

$$T = 4t_Q = 10{\cdot}44288294\ \text{s}. \tag{30}$$

The interval of the concatenated displacement corresponding to the first quarter cycle will now be reflected with respect to the line $t = t_Q$ in order to obtain the half-cycle. This half-cycle will now be repeated. All this is accomplished simply by first substituting the value of T from equation (30) into equation (10) and then replacing t by $R = R(t, T)$ in equation (26):

$$x_C(R) = x_1(R) + H(R, t_e)(-x_1(R) + x_2(R)), \tag{31}$$

or, equivalently, replacing t by R in equation (27):

$$x_C(R) = v(0)R + H(R, t_e)\left[-v(0)R + e + v(0)\sqrt{\frac{m}{k}}\sin\left\{\sqrt{\frac{k}{m}}(R - t_e)\right\}\right]. \tag{32}$$

$x_C(R)$, Figure 4(c), was plotted directly from equation (32).

It is now only necessary to convert the function of Figure 4(c) into an alternating function. First, the values (30) of T and $t_0 = 0$ are substituted into the alternator function, A; equation (13). Then, in accordance with equation (14) the product Ax_C is formed to yield the single equation representing the displacement–time solution

$$x(t) = A(t, T, t_0)\, x_C(R(t, T)), \tag{33}$$

x, Figure 4(d), was plotted directly from equation (33). In order to obtain the velocity–time relation for the first-quarter cycle, each of equations (23) and (24) is differentiated and then concatenated:

$$v_C(t) = v(0) + H(t, t_e)\left[-v(0) + v(0)\cos\left\{\sqrt{\frac{k}{m}}(t - t_e)\right\}\right]. \tag{34}$$

Note that the same result is obtained by differentiating equation (27) maintaining $H(t, t_e)$ constant. Resorting again to equation (14) the complete velocity–time solution

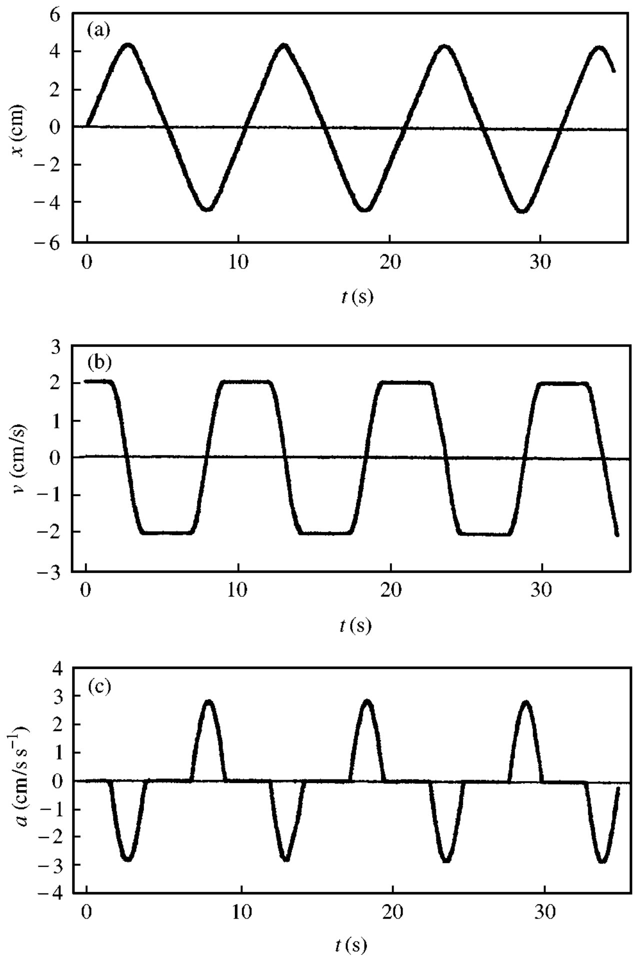


Figure 5. Example 1B: (a) displacement–time curve, equation (27); (b) velocity–time curve, equation (35); (c) acceleration–time curve, equation (37). The graphs were plotted directly from these equations after substitution of values (20) and (25).

is obtained:

$$v(t) = Av_C(R). \tag{35}$$

The acceleration–time relation for the first-quarter cycle is obtained by differentiating equation (34):

$$a_C(t) = -H(t, t_e)v(0)\sqrt{\frac{k}{m}}\sin\left\{\sqrt{\frac{k}{m}}(t - t_e)\right\}. \tag{36}$$

Applying equation (14) the complete acceleration–time solution is obtained:

$$a(t) = Aa_C(R). \tag{37}$$

The displacement, velocity and acceleration versus time graphs are obtained by directly plotting equations (33), (35) and (37) in Figure 5(a), 5(b) and 5(c) respectively.

4. FORCED, DAMPED VIBRATIONS

4.1. STEADY STATE, PROCEDURE

The basic idea is to obtain the piecewise general solution, i.e., both the particular integral and the complementary function, and to evaluate the integration constants as well as the times that define the "pieces" or intervals so as to satisfy both periodicity and analytical continuation requirements. Also, it is necessary to use the fact that the period of the steady state vibrations is the same as that of the excitation.

Consider a vibrating system containing a piecewise linear spring with n intervals each with a given different linear force–deflection relation. Suppose that the general piecewise solutions are: $x_1, x_2, \dots, x_n$, each containing two constants of integration still to be evaluated. If the deflection values at the extreme points of the intervals are given as $X_0, X_1, X_2, \dots, X_n$, the ranges of validity of the solutions are

$$X_0 \leqslant x_1 \leqslant X_1, \qquad X_1 \leqslant x_2 \leqslant X_2, \qquad X_2 \leqslant x_3 \leqslant X_3, \qquad \dots, \qquad X_{n-1} \leqslant x_n \leqslant X_n. \tag{38}$$

Corresponding to each X_i there is a time t_i also to be evaluated. Thus, there are $2n$ unknown constants of integration plus $n+1$ unknown values of the times at the extreme points of the intervals making a total of $3n+1$ unknowns.

There are four conditions to satisfy the periodicity requirements:

$$x_1(t_0) = X_0, \qquad x_n(t_n) = X_0, \qquad \dot{x}_1(t_0) = \dot{x}_n(t_n), \qquad t_n - t_0 = T. \tag{39–42}$$

There are $2(n-1)$ conditions to satisfy the analytical continuation requirements referring to displacement:

$$x_1(t_1) = X_1, \qquad x_2(t_1) = X_1, \qquad x_2(t_2) = X_2, \qquad x_3(t_2) = X_2, \qquad \dots,$$
$$x_{n-1}(t_{n-1}) = X_{n-1}, \qquad x_n(t_{n-1}) = X_{n-1}. \tag{43}$$

There are $n-1$ conditions to satisfy the analytical continuation requirements referring to velocity:

$$\dot{x}_1(t_1) = \dot{x}_2(t_1), \qquad \dot{x}_2(t_2) = \dot{x}_3(t_2), \qquad \dots, \qquad \dot{x}_{n-1}(t_{n-1}) = \dot{x}_n(t_{n-1}). \tag{44}$$

Thus, there is a total of $3n+1$ conditions which is the same as the total number of unknowns.

4.1.1. *Example* 2

The following data apply to the forced damped vibrating system of Figure 6 with preloaded springs: $m = 2$ N s^2/cm, $c = 0{\cdot}8$ N s/cm, $k = 2$ N/cm,

$$F_0 = 20 \text{ N}, \omega = 0{\cdot}22 \text{ s}^{-1}, \quad F_p = 4 \text{ N}, \tag{45}$$

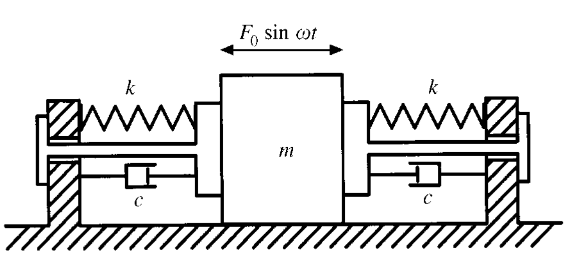


Figure 6. Example 2: forced damped vibrating system with preloaded springs.

where F_0 is the amplitude of the excitation force, ω is the excitation frequency and F_P is the magnitude of the preload. The steady state displacement, velocity and acceleration response will be obtained.

The displacement response will be obtained in piecewise fashion and afterwards it will be converted into a single equation. The discontinuous spring force versus displacement relation is represented graphically in Figure 7.

There are two distinct "pieces" or intervals with different spring force–deflection relations, i.e., $n = 2$; x_1 represents the positive displacement and x_2 the negative displacement. As mentioned above, in the steady state, there are four conditions associated with the periodicity requirements:

$$x_1(t_0) = 0, \qquad x_2(t_2) = 0, \qquad \dot{x}_1(t_0) = \dot{x}_2(t_2), \qquad t_2 - t_0 = T. \tag{46}$$

There are $3(n-1) = 3$ conditions associated with the analytical continuation requirements:

$$x_1(t_1) = 0, \qquad x_2(t_1) = 0, \qquad \dot{x}_1(t_1) = \dot{x}_2(t_1). \tag{47}$$

There are $3n + 1 = 7$ unknowns: the two constants of integration associated with each of x_1 and x_2 as well as the 3 times, t_0, t_1 and t_2. The problem may be solved handling all the mentioned quantities as unknowns; however, it will be simplified presently.

One period of the unknown displacement response will be expected to have the general characteristics shown in Figure 8. Due to the symmetrical arrangement of the mechanical

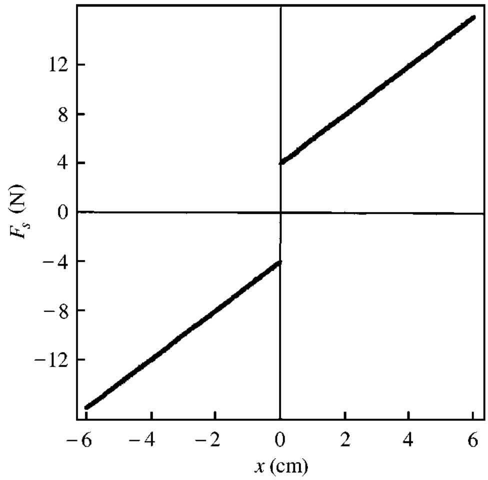


Figure 7. Example 2: spring force versus displacement. The discontinuity at $x = 0$ is due to the preloading of the springs. This graph was plotted directly from the single equation (67) after substituting values (45).

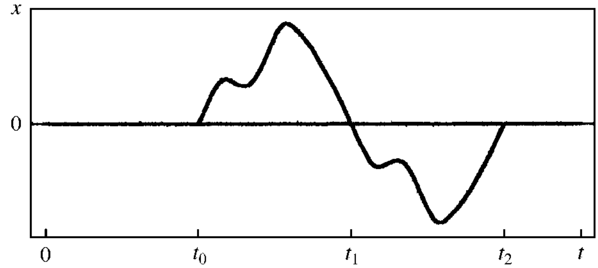


Figure 8. Example 2: expected general form of the displacement response cycle considering the symmetry of the mechanical arrangement.

elements the shape of the negative half-cycle displacement curve x_2, from t_1 to t_2 must be the same as the positive displacement curve x_1 from t_0 to t_1. In view of this only one interval need be considered; consequently, $n = 1$ and the seven conditions of equations (46) and (47) are reduced to the following four:

$$x_1(t_0) = 0, \qquad x_1(t_1) = 0, \qquad \dot{x}_1(t_0) = -\dot{x}_1(t_1), \qquad t_1 - t_0 = T/2. \tag{48}$$

There are now $3n + 1 = 4$ unknowns: the two constants of integration associated with x_1 as well as t_0 and t_1.

4.1.1.1. *First half-cycle*, $t_0 \leqslant t \leqslant t_1$. The differential equation of motion for this interval is

$$\ddot{x}_1 + \frac{c}{m}\dot{x}_1 + \frac{k}{m}x_1 = \frac{F_p}{m} + \frac{F_0}{m}\sin\omega t. \tag{49}$$

The general solution of this differential equation is

$$x_1(t) = \mathrm{e}^{-\zeta\omega_n t}(C_1\cos\omega_d t + C_2\sin\omega_d t) + A + B\sin\omega t + C\cos\omega t, \tag{50}$$

where

$$\begin{aligned} A &= -\frac{F_p}{k} = -\frac{F_p}{m\omega_n^2}, \\ B &= \frac{F_0(k - m\omega^2)}{(c\omega)^2 + (k - m\omega^2)^2} = \frac{F_0(\omega_n^2 - \omega^2)}{m\{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\}}, \\ C &= -\frac{F_0 c\omega}{(c\omega)^2 + (k - m\omega^2)^2} = -\frac{2F_0\zeta\omega_n\omega}{m\{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\}}. \end{aligned} \tag{51}$$

The steady state frequency and period are the same as those of the excitation:

$$\omega = 0{\cdot}22\ \mathrm{s}^{-1}, \quad T = 2\pi/\omega = 28{\cdot}559933\ \mathrm{s}. \tag{52}$$

Using the last of equations (48), C_1, C_2 and t_0 are determined as follows. The first two conditions (48) lead to the two equations

$$C_2 = \frac{\mathrm{e}^{\zeta\omega_n t_0}\cos\omega_d t_1(F_p/k - B\sin\omega t_0 - C\cos\omega t_0) + \mathrm{e}^{\zeta\omega_n t_1}\cos\omega_d t_0(-F_p/k + B\sin\omega t_1 + C\cos\omega t_1)}{k\sin\omega_d(t_0 - t_1)}, \tag{53}$$

$$C_1 = -\frac{\mathrm{e}^{\zeta\omega_n t_0}(-F_p/k + B\sin\omega t_0 + C\cos\omega t_0) + C_2\sin\omega_d t_0}{\cos\omega_d t_0}. \tag{54}$$

The third condition (48) yields

$$\begin{aligned} \varepsilon = {} & \mathrm{e}^{-\zeta\omega_n t_0}\{(-\zeta\omega_n C_1 + \omega_d C_2)\cos\omega_d t_0 + (-\zeta\omega_n C_2 - \omega_d C_1)\sin\omega_d t_0\} \\ & + B\omega\cos\omega t_0 - C\omega\sin\omega t_0 \\ & + \mathrm{e}^{-\zeta\omega_n t_1}\{(-\zeta\omega_n C_1 + \omega_d C_2)\cos\omega_d t_1 + (-\zeta\omega_n C_2 - \omega_d C_1)\sin\omega_d t_1\} \\ & + B\omega\cos\omega t_1 - C\omega\sin\omega t_1. \end{aligned} \tag{55}$$

In the previous equation ε should be equal to zero but, since it is impossible to solve for t_0 analytically, ε is considered a residue. Thus, substituting the equations given in Appendix B (referring to nomenclature) for ω_n, ω_d and ζ as well as numerical values (45) and then minimizing ε^2 yields the following values for the four unknowns:

$$t_0 = -0{\cdot}432909, \qquad t_1 = 13{\cdot}847058, \qquad C_1 = 3{\cdot}69438, \qquad C_2 = -0{\cdot}620649. \quad (56)$$

Values (56) are considered exact since they correspond to a value of ε in the order of 10^{-7}.

In view of the symmetrical mechanical arrangement the displacement in the second half-cycle is

$$x_2(t) = -x_1(t-0{\cdot}5T) = -\mathrm{e}^{-\zeta\omega_n(t-0{\cdot}5T)}\{C_1\cos\omega_d(t-0{\cdot}5T) + C_2\sin\omega_d(t-0{\cdot}5T)\}$$
$$+ A + B\sin\omega(t-0{\cdot}5T) + C\cos\omega(t-0{\cdot}5T). \quad (57)$$

The third half-cycle is

$$x_3(t) = x_1(t-T) = -\mathrm{e}^{-\zeta\omega_n(t-T)}\{C_1\cos\omega_d(t-T) + C_2\sin\omega_d(t-T)\}$$
$$+ A + B\sin\omega(t-T) + C\cos\omega(t-T). \quad (58)$$

Concatenating the first three half-cycles in accordance with equation (7):

$$x_C(t) = x_1(t) + H(t,t_1)\{-x_1(t) + x_2(t)\} + H(t,t_2)\{-x_2(t) + x_3(t)\}. \quad (59)$$

The concatenated displacement equation is now periodized:

$$x(p) = x_1(p) + H(p,t_1)\{-x_1(p) + x_2(p)\} + H(p,t_2)\{-x_2(p) + x_3(p)\}. \quad (60)$$

The period, T, of p is the period of the excitation: the second of equation (52). Equation (60) is the single, closed-form equation representing the displacement response at any time t. Figure 9(b) is a direct plot of this equation.

It must be emphasized that the periodizer function reproduces indefinitely the displacement response it finds in its own first period, which always starts at $t = 0$ (Figure 1) and, in this case, ends at $t = T = 28{\cdot}5599$ s. However, the first two half-cycles of the displacement response occur from $t = t_0 = -0{\cdot}4329$ to $t = t_0 + T = 28{\cdot}1270\ \mathrm{s} < T$. Thus, the two half-cycles are not enough for a correct reproduction and this is why it was necessary to include the third half-cycle.

Differentiating equation (47) yields the velocity for the first half-cycle:

$$\dot{x}_1(t) = \mathrm{e}^{-\zeta\omega t}\{(-\zeta\omega_n C_1 + \omega_d C_2)\cos\omega_d t + (-\zeta\omega_n C_2 - \omega_d C_1)\sin\omega_d t\}$$
$$+ B\omega\cos\omega t - C\omega\sin\omega t. \quad (61)$$

In the same manner as with the displacements, the second and third half-cycles of the velocity are obtained by performing the operations

$$\dot{x}_2(t) = -\dot{x}_1(t-0{\cdot}5T), \qquad \dot{x}_3(t) = \dot{x}_1(t-T). \quad (62)$$

Proceeding exactly as with the displacement, the single, closed-form velocity equation is obtained:

$$\dot{x}(p) = \dot{x}_1(p) + H(p,t_1)\{-\dot{x}_1(p) + \dot{x}_2(p)\} + H(p,t_2)\{-\dot{x}_2(p) + \dot{x}_3(p)\}. \quad (63)$$

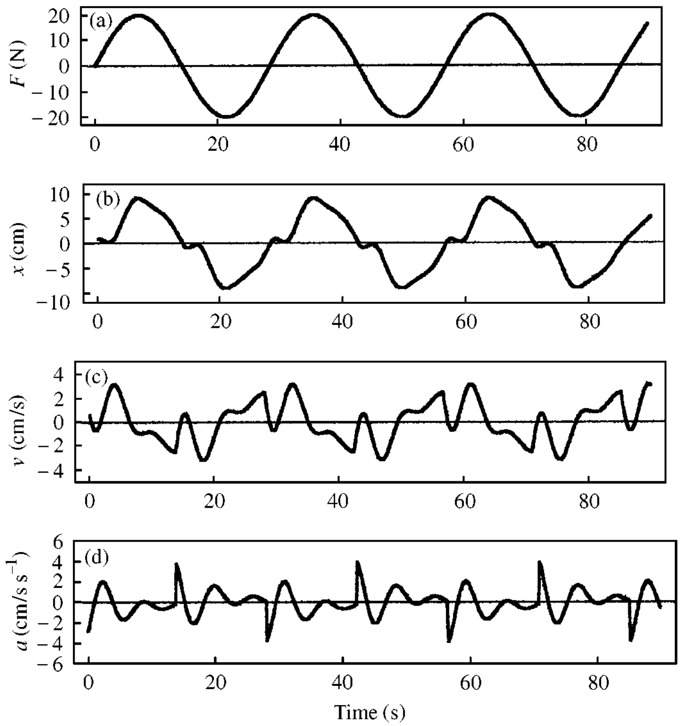


Figure 9. Example 2: (a) excitation force; (b) displacement response, direct plot of equation (60); (c) velocity response, direct plot of equation (63); (d) acceleration response, direct plot of equation (65).

Differentiating equation (61) yields the acceleration response for the first half-cycle:

$$\ddot{x}(t) = e^{-\zeta\omega_n t}[\{(\zeta^2\omega_n^2 - \omega_d^2)C_1 - 2\zeta\omega_n\omega_d C_2\}\cos\omega_d t + \{(\zeta^2\omega_n^2 - \omega_d^2)C_2 + 2\zeta\omega_n\omega_d C_1\}\sin\omega_d t] - B\omega^2\sin\omega t - C\omega^2\cos\omega t. \tag{64}$$

Proceeding just as in the case of the displacement and the velocity, the single, closed-form equation for the acceleration response is obtained:

$$\ddot{x}(p) = \ddot{x}_1(p) + H(p, t_1)\{-\ddot{x}_1(p) + \ddot{x}_2(p)\} + H(p, t_2)\{-\ddot{x}_2(p) + \ddot{x}_3(p)\}. \tag{65}$$

Figure 9(b), 9(c) and 9(d) are direct plots of equations (60), (63) and (65) respectively.

4.1.1.2. *Verification.* Resorting to equation (7), the spring force as a function of the displacement may be represented by

$$F_s = kx - F_p + H(x, 0)\{-(kx - F_p) + (kx + F_p)\}. \tag{66}$$

Making use of equation (6) to simplify equation (66) yields

$$F_s = kx + J(x, 0)F_p. \tag{67}$$

As a matter of fact, Figure 7 is a direct plot of equation (67). Thus, the differential equation of motion may be represented by

$$m\ddot{x} + c\dot{x} + kx + J(x, 0)F_p = F_0 \sin\omega t. \tag{68}$$

Substituting equations (60), (63) and (65) into the left-hand side of equation (68) and plotting yields the right-hand side (Figure 9(a)).

4.1.1.3. *Equivalent excitation.* Now that t_0 is known, equation (68) may be expressed as

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t - A(t, T, t_0)F_p. \tag{69}$$

It is interesting to note that equation (69) may be considered as referring to a linear vibration problem with plain, non-preloaded springs with an equivalent excitation represented by the right-hand member of the equation.

5. CONCLUSIONS

A methodology has been proposed to obtain the closed-form solutions of vibration problems involving piecewise linear springs. These solutions refer to free, undamped vibrations and to the steady state of forced, damped vibrations.

The procedure proposed to obtain the phase-plane solution of the free undamped vibration problem involves unification into a single expression of all the "pieces" of the piecewise representation of the force–deflection relation of the spring. This unified expression is incorporated into the equation of motion which is then integrated analytically. Only one constant of integration is required independently of the number of "pieces" of the piecewise representation. The solution is a single equation and it is exact. Additionally, a procedure is proposed to convert the various piecewise equations of the displacement–time solution covering a quarter of a period into a single expression valid for any time.

The procedure proposed to obtain the steady state response of forced, damped systems makes use of the general piecewise solutions including both the particular integral and the complementary function. The constants of integration and the limiting times of the intervals that define the force–deflection relation of the springs are determined by the requirements of periodicity and analytical continuation. The resulting displacement–time response is a single, closed-form equation. Since the displacement response is exact the velocity and acceleration responses are obtained by differentiation.

ACKNOWLEDGMENTS

The author wishes to express his appreciation for the valuable suggestions of the reviewers. The computer drawings of Figures 2 and 6 are due to Carlos González Rodríguez.

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APPENDIX A: INTEGRATION

Consider the integral

$$\int x^N J(x, 0)\,\mathrm{d}x. \tag{A1}$$

Now, according to relations (4) and (5),

$$\int x^N J(x, 0)\,\mathrm{d}x = \int x^{N-1}\sqrt{x^2}\,\mathrm{d}x. \tag{A2}$$

The value of this integral will be obtained by resorting to the following formula, obtained from an ordinary table of integrals [8]:

$$\int x^m (a + bx^q)^p\,\mathrm{d}x = \frac{x^{m-q+1}(a + bx^q)^{p+1}}{b(qp + m + 1)} - \frac{a(m - q + 1)}{b(qp + m + 1)}\int x^{m-q}(a + bx^q)^p\,\mathrm{d}x. \tag{A3}$$

Evidently, formula (A3) applies to the right-hand member of equation (A2) for

$$a = 0, \qquad b = 1, \qquad m = N - 1, \qquad q = 2, \qquad p = 1/2,$$

thus

$$\int x^N J(x, 0)\,\mathrm{d}x = \frac{x^{N-2}(x^2)^{3/2}}{N + 1} = \frac{x^{N-2}x^2\sqrt{x^2}}{N + 1} \tag{A4}$$

and finally

$$\int x^N J(x, 0)\,\mathrm{d}x = \frac{x^{N+1}}{N + 1} J(x, 0), \qquad N \neq -1. \tag{A5}$$

The procedure used to obtain expression (A5) was shown in considerable detail to point out that $\sqrt{x^2}$ is kept distinct from x

Equation (8) follows from equations (5) and (A5).

APPENDIX B: NOMENCLATURE

$A = A(t) = A(t, T, t_0)$	alternator function, equations (11)–(13)
a	a constant or acceleration depending on context
c	damping coefficient
$c_{cr} = 2\sqrt{km}$	critical damping coefficient
F	force
F_0	amplitude of the forcing function

F_p	spring preload force
$H(x, a)$	Heaviside unit step function at point $x = a$, equation (6)
$J(x, a)$	jump or relay at point $x = a$, equation (5)
k	spring constant
m	mass
n	number of intervals with different spring force–deflection relations
$p = p(t) = p(t, T)$	periodizer function, equation (9)
$R = R(t) = R(t, T)$	reflecting and repeating function, equation (10)
t	time or independent variable
t_0	phase time
T	period
x	independent variable or displacement depending on context
x_i	displacement valid during the ith interval
v	velocity
$V(x, a)$	absolute value of $x - a$, equation (4)
ε	residue
$\zeta = c/c_{cr}$	damping factor
ϕ	phase angle
ω	frequency of excitation
$\omega_d = \omega_n\sqrt{1 - \zeta^2}$	damped frequency
$\omega_n = \sqrt{k/m}$	natural frequency
$(\ )_C$	concatenated
$(\ )_Q$	referring to one-quarter of a cycle
$(\dot{\ })$	first derivative with respect to time
$(\ddot{\ })$	second derivative with respect to time